The Post-Newtonian Maclaurin Spheroids to Arbitrary Order

David Petroff *

Sept. 3 2003

Abstract

In this paper, we develop an iterative scheme to enable the explicit calculation of an arbitrary post-Newtonian order for a relativistic body that reduces to the Maclaurin spheroid in the appropriate limit. This scheme allows for an analysis of the structure of the solution in the vicinity of bifurcation points along the Maclaurin sequence. The post-Newtonian expansion is solved explicitly to the fourth order and its accuracy and convergence are studied by comparing it to highly accurate numerical results.

1 Introduction

Upon the discovery of pulsars in 1968 and their identification as neutron stars, it became apparent that a relativistic description of rapidly rotating, compact stars was needed. Early work in this direction dealt with simplified models for the matter making up these objects. In particular, Chandrasekhar [1] looked at stars of constant density and calculated the first post-Newtonian correction to the Maclaurin spheroids. Bardeen [2] reexamined this work using a modified approach and gained new insight regarding, foremost, the points of onset of secular, axisymmetric instability along a one parameter Maclaurin curve.

Given the amount of work that has been done since then to study stars with more realistic equations of state, the return to a model of constant density can hardly be motivated by astrophysical considerations. Many other good arguments however, suggest that precisely this model deserves closer attention: For one, it allows, as we shall see, for the development of an iterative scheme to calculate explicitly any order of the post-Newtonian expansion, limited in practice only by computer algebra programs and

*D.Petroff@tpi.uni-jena.de

the machines running them. Furthermore, by considering an arbitrary order, one can study properties of the full relativistic solution and carry out Bardeen's task of testing conjectures "by going to higher orders in the relativistic expansion". Finally, due to the fortuitous circumstance of being in possession of highly accurate numerical values, one can go even further. We are in the rare position of being able to examine the behaviour of the post-Newtonian expansion itself, providing, by analogy, a testbed for the most widely used analytic approximation within the field of General Relativity.

In section 2 of this paper, we motivate the method used here by briefly describing Bardeen's approach [2] for the first order of the expansion and explaining why modifications are necessary when going to higher orders. Section 3 presents the line element and the Einstein equations to be solved for iteratively. An iterative scheme allowing for the explicit calculation of an arbitrary order is presented in section 4 and some properties of the solution are discussed in section 5. After providing by way of example the explicit calculation of a few expressions and introducing various physical quantities, the PN approximation up to the fourth order is compared with highly accurate numerical results in section 6.

2 Preliminary Remarks

For a given mass-density, Q, the Maclaurin spheroids and the relativistic model both depend on two parameters.[1] Since the post-Newtonian approximation describes the relativistic model in terms of Newtonian parameters, some convention is needed to determine which relativistic parameters are implied by the specification of the Newtonian ones. Bardeen argued that the "most appropriate choice" compares Newtonian and relativistic bodies of the same rest mass M_0 and angular momentum J since these quantities (together with Q) are coordinate independent and "play the primary role in the Hartle-Sharp [3] variation principle". In this paper we take a somewhat different approach since our purpose is less the comparison of Newtonian and relativistic configurations, than the development of a method for calculating an arbitrary order of the expansion. Therefore we use the freedom that one has in defining the PN approximation in order to simplify the mathematical structure of the equations. The remaining freedom regarding the choice of a constant is left unspecified as long as possible. What effect the specification of this constant then has, will be studied in section 6.3 of the paper.

At this point, a brief description of the method that Bardeen used in [2] will provide us with the basis for understanding the motivation for the methods used in this paper.[2] Up to the first order of the PN approximation, one has to determine two metric functions

[1] In the Newtonian case, one of these is a mere scaling parameter.

[2] The reader who is interested in studying Bardeen's paper [2] may benefit from the following list of errata. The coefficient of $P_2(\eta)$ in eq. (21) should read $\frac{1}{p_2(\xi_s)}\left[\frac{1}{2}p_2(\xi)C_2+\frac{{\xi_s}^2-\xi^2}{\xi_s(1+\xi_s)^{1/2}}\right]$. The left hand side of eq. (36) should read $\frac{1}{c^2}(v^2)_E(\xi_s,\eta)$ and on the right hand side $\frac{1}{12}D_2W_2(\eta)$. The $P_l(\eta)$ of eq. (43)

from Poisson-like equations as well as the unknown boundary of the star. To solve for the metric functions, Bardeen used a Poisson-integral in spheroidal coordinates ξ and η, which represents potentials as an expansion in terms of orthogonal polynomials in η. An iterative scheme for the calculation of higher orders is only feasible if the sum over these polynomials terminates. The conditions for the termination of the sum are that the source remain a polynomial in η and that the boundary of the star remain a constant in ξ. Neither of these conditions is met with in Bardeen's approach, which is why it is only appropriate up to the first order. To that order it was possible to determine the metric functions in an elegant way, because they can be decomposed into one piece containing the new (post-Newtonian) source within the old boundary and another piece containing the old source within the new boundary. For higher orders, such a procedure can no longer be used and one has to devise a modified approach.

The approach used in this paper relies on the fact that an extended version of the Poisson-integral is valid for Poisson-like equations even in modified coordinates. Here coordinates will be introduced that are tailored to the unknown boundary of the star and satisfy the condition that the boundary be a constant in this coordinate. Furthermore we require that the unknown boundary of the star when written as a function of the old coordinate ξ be given as a polynomial in η, a requirement that can be shown to be compatible with the condition that the pressure vanish at the surface of the star. This requirement ensures that the sources in the Poisson-like equations remain polynomials in η. Thus we have to deal only with terminating sums to any order of the PN approximation and the recursive method proposed here can be applied indefinitely.

3 Basic Equations

The line element for an axially symmetric, stationary, asymptotically flat space-time describing a perfect fluid with purely azimuthal motion can be written in Lewis-Papapetrou coordinates as

$$ds^2 = e^{2\mu}\left(d\varrho^2 + d\zeta^2\right) + \varrho^2 e^{2\lambda}\left(d\varphi - \omega\, dt\right)^2 - c^2 e^{2\nu} dt^2.$$

The metric functions μ, λ, ω and ν depend only on ϱ and ζ and vanish at spatial infinity. The energy-momentum tensor for the pressure P and the mass density Q, which is merely the energy density divided by c^2, is then given by

$$T_{\alpha\beta} = \left(Qc^2 + P\right) u_\alpha u_\beta + g_{\alpha\beta} P,$$

where Q is a constant up to the surface of the star. The matter of the star rotates uniformly with an angular velocity Ω. We introduce the spheroidal coordinates

$$\varrho^2 = {a_0}^2(1+\xi^2)(1-\eta^2) \quad \text{and} \quad \zeta = a_0\xi\eta, \quad \eta \in [-1,1], \quad \xi \in [0,\infty),$$

is to be replaced by $(P_l(\eta) - 1)$ and the second term in the first line of eq. (55) by $+\frac{\Delta\xi}{c^2}\frac{\partial U}{\partial \xi}(\xi_s, 1)$.

and obtain from various combinations of the Einstein equations the following partial differential equations for the metric functions (with $G = 1$ for the gravitational constant):

$$\begin{aligned}\Delta_2\nu = {} & \frac{4\pi e^{2\mu}}{c^4}\left[\frac{1+\tilde{v}^2}{1-\tilde{v}^2}(Qc^2+P)+2P\right] - \mathcal{L}(\nu,\nu+\lambda) \\ & + \frac{1}{2}\tilde{\Omega}^2(1+\xi^2)(1-\eta^2)e^{2\lambda-2\nu}\mathcal{L}(\tilde{\omega},\tilde{\omega}),\end{aligned} \tag{1a}$$

$$\Delta_3(\lambda+\nu) = \frac{16\pi e^{2\mu}}{c^4}P - \mathcal{L}(\nu+\lambda,\nu+\lambda), \tag{1b}$$

$$\Delta_4\tilde{\omega} = \frac{-16\pi(1-\tilde{\omega})e^{2\mu}}{c^4(1-\tilde{v}^2)}(Qc^2+P) - \mathcal{L}(\tilde{\omega},3\lambda-\nu), \tag{1c}$$

$$\begin{aligned}\Delta_1\mu = {} & \frac{-4\pi e^{2\mu}}{c^4}(Qc^2+P) + \mathcal{L}(\nu,\lambda) + \frac{1}{4}(1+\xi^2)(1-\eta^2)e^{2\lambda-2\nu}\mathcal{L}(\tilde{\omega},\tilde{\omega}) \\ & + \frac{1}{a_0^2(\xi^2+\eta^2)}\left(\xi\nu_{,\xi} - \eta\nu_{,\eta}\right).\end{aligned} \tag{1d}$$

The differential operators in the above equations are defined by

$$\mathcal{L}(\phi,\chi) := \left[(1+\xi^2)\phi_{,\xi}\chi_{,\xi} + (1-\eta^2)\phi_{,\eta}\chi_{,\eta}\right]/a_0^2(\xi^2+\eta^2)$$

and

$$\Delta_m := \left[(1+\xi^2)\frac{\partial^2}{\partial\xi^2} + (1-\eta^2)\frac{\partial^2}{\partial\eta^2} + m\xi\frac{\partial}{\partial\xi} - m\eta\frac{\partial}{\partial\eta}\right]/a_0^2(\xi^2+\eta^2)$$

and the dimensionless function in eq. (1c) by

$$\tilde{\omega} := \frac{\omega}{\Omega}.$$

Note that the operator Δ_2 is simply the Laplace operator in a flat three-dimensional space. The dimensionless pressure $\tilde{P} := P/Qc^2$ is related to the metric functions by

$$\sqrt{1+\tilde{v}^2}\,(1+\tilde{P})\,e^{\nu} = \text{const.} = 1-\gamma \tag{2}$$

with

$$\tilde{v} := \varrho\tilde{\Omega}(1-\tilde{\omega})e^{\lambda-\nu}/a_0 \quad \text{and} \quad \tilde{\Omega} := a_0\Omega/c.$$

4 The Iterative Scheme

4.1 The Expansion

The system of partial differential equations (1) is simplified by expanding the relevant quantities in terms of a dimensionless relativistic parameter. Here we choose the square root of the parameter[3] used in [2] defined by

$$\varepsilon^2 := \frac{8\pi Q {a_0}^2 \xi_s \sqrt{1+{\xi_s}^2}}{3c^2}. \tag{3}$$

The three variables entering into this definition completely specify the Newtonian Maclaurin spheroid. Q is the mass density, a_0 the focus of the ellipse describing the surface of the star in cross-section and ξ_s the value of the surface's ξ coordinate. These are the same quantities which will enter into the PN expansion, but the latter two lose their simple geometrical meaning. The parameter ε remains finite in both the spherical limit, given by $\xi_s \to \infty$ and $a_0 \propto 1/\xi_s$ and the disc limit, which is given by $\xi_s \to 0$ and $Q \propto 1/\xi_s$ for non-vanishing mass.

The expansion of the dimensionless metric functions and the constants reads as follows:

$$\begin{aligned} \nu &= \sum_{n=2}^{\infty} \nu_n \varepsilon^n \quad & \lambda &= \sum_{n=2}^{\infty} \lambda_n \varepsilon^n \quad & \tilde{\omega} &= \sum_{n=2}^{\infty} \tilde{\omega}_n \varepsilon^n \\ \mu &= \sum_{n=2}^{\infty} \mu_n \varepsilon^n \quad & \gamma &= \sum_{n=2}^{\infty} \gamma_n \varepsilon^n \quad & \tilde{\Omega} &= \sum_{n=1}^{\infty} \tilde{\Omega}_n \varepsilon^n. \end{aligned} \tag{4}$$

As was already mentioned, Q is held constant to any order of the approximation, which is why it does not appear in eq. (4) and any other quantities of interest, such as $\tilde{v}$ or $\tilde{P}$ can be expressed in terms of these six quantities.

If these expansions are substituted into eqs (1), then comparing coefficients of ε yields differential equations for the metric functions of the form $\Delta_m \phi_i = F$, where $\phi = \nu, \lambda, \tilde{\omega}, \mu$. Because the right hand side of eqs (1a–1c) depends only on ϕ_{i-j}, $j > 0$, one can solve for ϕ_i if the lower order functions are already known. In the case of μ_i, one can calculate it from eq. (1d) after having determined the other three functions to this order, or one can compute it from an integral over η.

Because an analytic solution, the Maclaurin solution, for the first step is known, these equations would provide an iterative process for the determination of the metric functions to any order if the shape of the star were known. The boundary of the star also has to be determined iteratively however.

[3] The square root was chosen in order to enable a more convenient indexing of the expansion coefficients.

We represent the surface of the star by the equation,

$$\xi = \xi_{\mathrm{B}}(\eta) = \xi_{\mathrm{s}} \left(1 + \sum_{j=0}^{k} \sum_{k=2}^{\infty} S_{jk}\, C_j^{1/2}(\eta)\, \varepsilon^k \right) \equiv \xi_{\mathrm{s}} \left(1 + \sum_{k=2}^{\infty} B_k(\eta) \varepsilon^k \right), \tag{5}$$

where we have already taken into account that the boundary is an ellipsoid $\xi = \xi_{\mathrm{s}}$ in the Newtonian order. We also require that the sum over the Legendre polynomials $C_j^{1/2}(\eta)$, a special case of the Gegenbauer polynomials discussed in section 4.2, terminate and show in section 4.4 that this leads to a consistent solution.

4.2 Solving the Poisson-like Equations

In the last section an iterative scheme was proposed for the determination of the metric functions in which an equation of the form $\Delta_m \phi = F$ need be solved for a known function $F = F(\xi, \eta)$. The regular and asymptotically flat solution of this equation is given by

$$\begin{aligned} \phi(\xi,\eta) = {} & a_0^2 \sum_{l=0}^{\infty} K_l^m C_l^{\frac{m-1}{2}}(\eta) \times \\ & \Big[h_l^m(\xi) \int_0^{\xi} \int_{-1}^{1} g_l^m(\xi') C_l^{\frac{m-1}{2}}(\eta') F(\xi',\eta') k_m(\xi',\eta')\, d\eta'\, d\xi' + \\ & g_l^m(\xi) \int_{\xi}^{\infty} \int_{-1}^{1} h_l^m(\xi') C_l^{\frac{m-1}{2}}(\eta') F(\xi',\eta') k_m(\xi',\eta')\, d\eta'\, d\xi' \Big]. \end{aligned} \tag{6}$$

In the above equation C_i^j are the Gegenbauer polynomials, g_i^j and h_i^j are two linearly independent solutions of the (homogeneous) Gegenbauer equation defined by

$$\begin{aligned} g_l^m(\xi) &:= C_l^{\frac{m-1}{2}}(i\xi) \\ h_l^m(\xi) &:= g_l^m(\xi) \int_{\xi}^{\infty} \frac{d\xi'}{\left(g_l^m(\xi')\right)^2 E(\xi')} \qquad (l,m) \neq (0,1) \quad \text{and} \\ h_0^1(\xi) &:= \operatorname{arcsinh}(\xi) \end{aligned} \tag{7}$$

with

$$E(\xi) := \exp\left(\int_0^{\xi} \frac{m\xi'}{1+\xi'^2}\, d\xi' \right) = \left(1+\xi^2\right)^{m/2}.$$

The term

$$k_m(\xi,\eta)\,d\eta\,d\xi := \left[\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{\frac{m}{2}-1}\left(\xi^2+\eta^2\right)d\eta\,d\xi \tag{8}$$

is a product of the volume element and the appropriate weight function for the Gegenbauer polynomials and

$$\begin{aligned} K_l^m &= \frac{l!\left(l+\frac{m}{2}-\frac{1}{2}\right)\left[\Gamma\left(\frac{m}{2}-\frac{1}{2}\right)\right]^2}{\pi 2^{2-m}\Gamma(m-1+l)} \quad \text{for } m>1 \\ K_l^1 &= \frac{l^2}{2\pi} \quad \text{for } l>0 \quad \text{and} \\ K_0^1 &= \frac{1}{\pi} \end{aligned}$$

are normalizing constants.

In eq. (6) the integrands jump at the surface of the star because of the jump in the mass density. It is therefore necessary to split them into integrals over the interior and exterior of the star. Clearly if the surface of the star is given, as with the leading order, by a constant $\xi = \xi_{\rm s}$, then this division is trivial. If the boundary depends on η, then matters are complicated considerably. As of the second order in the expansion, the η-integrals no longer run over the interval $\eta \in [-1,1]$ meaning that one can no longer make use of the orthogonality of the Gegenbauer polynomials and one is faced with non-terminating sums. We hence introduce new coordinates in order to circumvent these difficulties.

4.3 New Coordinates

We introduce the coordinates

$$\psi = \frac{\xi_{\rm s}\xi}{\xi_B(\eta)}, \quad \eta = \eta \tag{9}$$

implying that $\psi = \xi_{\rm s}$ is the boundary of the star, cf. eq. (5). The new coordinate ψ is a function of both η and ε and contains the unknown coefficients S_{jk}. We rewrite eqs (1) in terms of the new coordinates and manipulate them such that the left hand side has the same form as beforehand, but with ψ replacing ξ. For example, the equation for $\nu = \nu(\psi,\eta)$ now reads

$$\left[(1+\psi^2)\frac{\partial^2\nu}{\partial\psi^2} + (1-\eta^2)\frac{\partial^2\nu}{\partial\eta^2} + 2\psi\frac{\partial\nu}{\partial\psi} - 2\eta\frac{\partial\nu}{\partial\eta}\right]/a_0^2(\psi^2+\eta^2) = \bar{F}.$$

These new field equations are again expanded[4] in terms of ε in order to obtain a system of equations for ϕ_i as was explained in section 4.1. Since $\psi = \xi + \mathcal{O}(\varepsilon^2)$ holds, the

[4]Note that the coefficients $\phi_i(\psi,\eta)$ are not mere transformations of $\phi_i(\xi,\eta)$ since ψ depends on ε. Thus one must substitute ψ of eq. (9) into $\sum_{j=2}^{n}\phi_j(\psi,\eta)$ and expand the result in terms of ε in order to express ϕ_k, $k \le n$, in terms of ξ.

new equations for $\phi_i(\psi, \eta)$ also depend only on known functions, thereby enabling their recursive determination.

The derivation of eq. (6) relies on the fact that in the coordinates (ξ, η), the line $(0, \eta)$ is identical to the line $(0, -\eta)$ and that at spatial infinity we have $\xi \to \infty$. These properties hold for ψ as well and an analysis of the derivation shows that we are free to use eq. (6) as it stands, only replacing ξ by ψ.

In changing coordinates we have mapped the star onto the rectangle $[0, \xi_\mathrm{s}] \times [-1, 1]$, which means that the division of the integrals into inner and outer domains is trivial. The price that one pays for the simplicity in the structure of the integrals is that the sources of the Poisson-like equations become quite unwieldy. But the exchange of a conceptual for a mechanical difficulty can be termed a good deal, and all the more so when its result is the facilitation of the whole scheme.

4.4 Determining the Shape of the Star

Due to the factor c^2 in g_{tt} of the line element, it is necessary to determine the function ν_{i+2} in order to calculate the i^{th} order of the PN approximation. This is the only metric function that depends on the unknown coefficients S_{ji} of the star's boundary.[5] To determine these coefficients, one calculates the pressure from eq. (2) along the boundary of the star and sets the coefficients of an expansion in terms of η equal to zero. In discussing the boundary, however, it turns out to be useful to leave a portion of the Poisson-integral for ν_{i+2} unevaluated in order to arrive at the integral equation

$$\tilde{\Omega}_1^2(1-\eta^2)\,\xi_\mathrm{s}{}^2\,B_i(\eta) = \sum_{l=0}^{\infty} K_l^2\, C_l^{1/2}(\eta) \int_{-1}^{1} C_l^{1/2}(\tilde{\eta}) f_l(\tilde{\eta}) B_i(\tilde{\eta})\, d\tilde{\eta} + b_i(\eta). \tag{10}$$

The function B_i is defined in eq. (5) and contains the entire dependence on S_{ji}. The function $b_i(\eta)$ is short for the remaining terms that result from eq. (2) and is a known polynomial of order $i+2$. The function $f_l(\tilde{\eta})$ is given by

$$\begin{aligned} f_l(\tilde{\eta}) :=& g_l^2(\xi_\mathrm{s}) \int_{\xi_\mathrm{s}}^{\infty} h_l^2(\psi) \Big[2\tilde{\eta}\psi(\dot{\nu}_2^\mathrm{o})' - \psi(1-\tilde{\eta}^2)(\dot{\nu}_2^\mathrm{o})'' + 2\ddot{\nu}_2^\mathrm{o} - l(l+1)\psi\dot{\nu}_2^\mathrm{o} \Big]\, d\psi \\ &+ h_l^2(\xi_\mathrm{s}) \int_0^{\xi_\mathrm{s}} g_l^2(\psi) \Big[2\tilde{\eta}\psi(\dot{\nu}_2^\mathrm{i})' - \psi(1-\tilde{\eta}^2)(\dot{\nu}_2^\mathrm{i})'' + 2\ddot{\nu}_2^\mathrm{i} + \frac{3\psi^2}{\xi_\mathrm{s}\sqrt{1+\xi_\mathrm{s}{}^2}} - l(l+1)\psi\dot{\nu}_2^\mathrm{i} \Big]\, d\psi, \end{aligned}$$

where a dot and prime indicate partial derivatives with respect to ψ and $\tilde{\eta}$ respectively and the superscripts 'i' and 'o' refer to the regions inside and outside the star. Since

[5] The other metric functions depend only on S_{jk}, with $j < i$.

$f_l(\eta)$ is a polynomial of second order, the sum in eq. (10) terminates for polynomial $B_i(\eta)$. Indeed, for the form of B_i chosen in eq. (5), one arrives at a system of $i+2$ algebraic equations for $i+3$ unknowns (there are $i+1$ S_{ij} to determine as well as $\tilde{\Omega}_{i+1}$ and γ_{i+2}).[6] We choose to use this system to determine all these constants but for γ_{i+2}. As mentioned in section 2, this last constant can be chosen arbitrarily, which amounts to specifying "which" PN-approximation one wishes to have, i.e. which relativistic body is to be associated with a given Maclaurin spheroid. The choice of γ_{i+2} will be discussed further in section 6.3.

We have shown that the form chosen for the surface of the star is consistent with the Einstein equations to any order of the PN expansion. This is not to say that this choice is unique. One can easily see that the form chosen in [2] is incompatible with that chosen here, since it is not a polynomial in η. There the surface was derived having stipulated that the 'generating' Maclaurin spheroid should have the same rest mass and angular momentum as the PN star, a condition that cannot be satisfied with the approach chosen here. In lieu of the freedom to choose two constants, we have chosen a form for the boundary of the star most appropriate to our goal of devising an iterative scheme and can choose only one further constant.

5 Properties of the Solution

5.1 Reflectional Symmetry

In Newtonian physics, it is known, that stationary, axisymmetric bodies are necessarily symmetric with respect to a reflection through the $\zeta = 0$ plane (see e.g. [4]). Although authors (e.g. [5]) have speculated that the same holds in General Relativity, it has not yet been proved true. In the case considered here, this symmetry arises automatically. A function f exhibits reflectional symmetry in ξ-η (or ψ-η) coordinates precisely when it is an even function of η. Because of the orthogonality of the Gegenbauer polynomials, the terms in the sum of eq. (6) for odd l are zero if F is a polynomial in η^2, a condition which turns out to be satisfied. The odd terms, which are provided by the unknown boundary coefficients S_{li} must be zero for the boundary condition to be fulfilled. Thus we have shown that any axially symmetric, stationary, relativistic solution that is continuously connected to the Maclaurin spheroids is symmetric with respect to reflections through the $\zeta = 0$ plane.

5.2 Powers of the Relativistic Parameter

Consideration of the field equations together with the knowledge of the Newtonian behaviour of the dimensionless metric functions, shows that their expansion coefficients

[6] We shall see in section 5.1 that $S_{ij} = 0$ for odd j.

l	ξ^*_{2l}	e^*_{2l}	$r_\mathrm{p}/r_\mathrm{e}$
2	0.17383011	0.98522554	0.17126187
3	0.11230482	0.99375285	0.11160323
4	0.08303471	0.99657034	0.08274493
5	0.06588682	0.99783651	0.06574427

Table 1: Numerical values for ξ^*_{2l} and the corresponding Newtonian eccentricities and ratios of polar to equatorial radii given by $e = 1/\sqrt{1+{\xi_\mathrm{s}}^2}$ and $r_\mathrm{p}/r_\mathrm{e} = \xi_\mathrm{s}/\sqrt{1+{\xi_\mathrm{s}}^2}$.

ϕ_i begin with $i=2$ and are non-zero only for even i. The same holds naturally for γ_i, whereas $\tilde{\Omega}_j$ begins with $j=1$ and appears only with odd powers powers of j. Because of the choice to work with the dimensionless functions introduced here, it is most appropriate to refer to the n^th order of the PN approximation and not the half orders in between. What we mean by the n^th order is that the quantities λ, $\tilde{\omega}$, μ and ξ_B are expanded up to and including the order $\mathcal{O}(\varepsilon^{2n})$, $\tilde{\Omega}$ up to $\mathcal{O}(\varepsilon^{2n+1})$ and ν and γ to $\mathcal{O}(\varepsilon^{2n+2})$.

5.3 Singularities in Parameter Space

By comparing the highest coefficient of η in eq. (10) one can arrive at the equation

$$S_{ii} = \frac{t_i}{{\xi_\mathrm{s}}^2\tilde{\Omega}_1^2 - a_{i+2}} \tag{11}$$

with

$$t_i \propto \int_{-1}^{1} C^{1/2}_{i+2}(\bar{\eta}) b_i(\bar{\eta})\, d\bar{\eta}$$

and

$$C^{1/2}_{i+2}(\eta) f_{i+2}(\eta) =: \sum_{n=0}^{1} \bar{a}_{2n,i+2}\eta^{2n}, \quad a_{i+2} := \bar{a}_{2,i+2}$$

and where f_{i+2} and b_i are defined in eq.(10). It can be shown that the denominator of eq. (11) is proportional to the expression

$$g^2_{i+2}(\xi_\mathrm{s}) h^2_{i+2}(\xi_\mathrm{s}) - \xi_\mathrm{s}(1 - \xi_\mathrm{s}\mathrm{arccot}(\xi_\mathrm{s})) =: G_i(\xi_\mathrm{s}). \tag{12}$$

For a given (even) i, this expression vanishes for precisely one value of ξ_s, let us say for $\xi_\mathrm{s} = \xi^*_{i+2}$. These values, beginning with $i=2$, are the points of onset of axisymmetric, secular instability and the bifurcation points of new axisymmetric solutions, see [2, 6, 7], and numerical values for the first few of them can be found in Table 1. Since t_i of eq. (11) is not zero at the point $\xi_\mathrm{s} = \xi^*_{i+2}$, these bifurcation points are singularities

in the two dimensional parameter space ($\xi_{\rm s}$,a_0) or ($\xi_{\rm s}$,ε). For values of $\xi_{\rm s}$ differing only slightly from ξ^*_{i+2}, the PN configurations have properties similar to those of the Newtonian configurations that branch off from the Maclaurin sequence at these points. The Maclaurin configuration itself cannot be reached for bodies with non-zero mass however, and even neighbouring configurations have strict mass limitations, since ε must be made very small in order that the PN series converge. This mass limitation can be inferred, for example, by referring to the tables in Appendix B. Because the n^{th} PN order possesses a pole of order $2n-1$ at the point $\xi = \xi^*_4$, we expect the coefficients for the expansion to grow large in the vicinity of this point.[7] This is indeed the case as can be seen in these tables by referring to the row with $\xi_{\rm s} = 0.17$. The series containing these coefficients converge only for sufficiently small ε as indicated above.

6 Explicit Solution to the Fourth Order

6.1 The Metric Functions and the Constants

Using the iterative scheme described above, the four metric functions and the constant $\tilde{\Omega}$ were explicitly solved up to the fourth post-Newtonian order. These calculations could in principle be carried out *ad infinitum*, but the lengthiness of the expressions (the fourth order functions would fill several hundred pages) puts a practical limit on the order that can be determined. Here we will merely carry out, by way of example, the calculation of the first few terms.

The expansion of eq. (1a) with respect to the relativistic parameter ε yields the Newtonian equations

$$\Delta_2 \nu_2^{\rm i} = \frac{4\pi}{\varepsilon^2 c^2} Q = \frac{3}{2a_0^2 \xi_{\rm s}\sqrt{1+\xi_{\rm s}^2}} \tag{13a}$$

for the interior region ($\xi < \xi_{\rm s}$) and

$$\Delta_2 \nu_2^{\rm o} = 0. \tag{13b}$$

for the region $\xi > \xi_{\rm s}$ exterior to the body. These equations are solved using eq. (6) to

[7] A lengthier discussion regarding the order of the poles at ξ^*_{2i+2}, $i > 1$ can be found in [8].

obtain

$$
\begin{aligned}
\nu_2^{\mathrm{i}} = & -\frac{1}{2\,\xi_{\mathrm{s}}\sqrt{1+{\xi_{\mathrm{s}}}^2}} \\
& \Bigg\{\Big[\xi_{\mathrm{s}}(1+{\xi_{\mathrm{s}}}^2)h_0^2(\xi_{\mathrm{s}})+\frac{1}{2}({\xi_{\mathrm{s}}}^2-\xi^2)\Big]C_0^{1/2}(\eta) \\
& +\Big[-\xi_{\mathrm{s}}\,(1+{\xi_{\mathrm{s}}}^2)\,g_2^2(\psi)\,h_0^2(\xi_{\mathrm{s}}) \\
& +\frac{1}{3}\left((3\,{\xi_{\mathrm{s}}}^2+2)g_2^2(\psi)+1\right)\Big]C_2^{1/2}(\eta)\Bigg\}
\end{aligned}
\tag{14a}
$$

and

$$
\nu_2^{\mathrm{o}} = \frac{-\sqrt{1+{\xi_{\mathrm{s}}}^2}}{2}\left(h_0^2(\psi)\,C_0^{1/2}(\eta)-h_2^2(\psi)\,C_2^{1/2}(\eta)\right) \tag{14b}
$$

(see Appendix A for a list of the first few g_l^m and h_l^m). One can verify that $\nu_2^{\mathrm{i}}(\xi_{\mathrm{s}},\eta) = \nu_2^{\mathrm{o}}(\xi_{\mathrm{s}},\eta)$ holds. The requirement that the pressure vanish to this order of the expansion then fixes the two remaining constants:

$$
\tilde{\Omega}_1 = -\frac{3}{2\sqrt{1+{\xi_{\mathrm{s}}}^2}}h_2^2(\xi_{\mathrm{s}}) \quad \text{and} \quad \gamma_2 = \frac{\sqrt{1+{\xi_{\mathrm{s}}}^2}}{2}\left(h_0^2(\xi_{\mathrm{s}})-h_2^2(\xi_{\mathrm{s}})\right). \tag{15}
$$

Expanding eqs (1b) and (1d) shows that $\lambda_2 = \mu_2 = -\nu_2$ holds and one need only expand eq. (1c) to obtain the last of the functions ϕ_2, where $\phi = \nu, \lambda, \tilde{\omega}, \mu$. This expansion leads to the equations

$$
\Delta_4\tilde{\omega}_2^{\mathrm{i}} = \frac{-6}{a_0^2\,\xi_{\mathrm{s}}\sqrt{1+\xi_{\mathrm{s}}^2}}
$$

and

$$
\Delta_4\tilde{\omega}_2^{\mathrm{o}} = 0,
$$

with the solutions

$$
\begin{aligned}
\tilde{\omega}_2^{\mathrm{i}} = & \frac{1}{5\,\xi_{\mathrm{s}}\sqrt{1+{\xi_{\mathrm{s}}}^2}} \\
& \Bigg\{\Big[6\xi_{\mathrm{s}}\left(1+{\xi_{\mathrm{s}}}^2\right)^2h_0^4(\xi_{\mathrm{s}})+3\,\left({\xi_{\mathrm{s}}}^2-\xi^2\right)\Big]C_0^{3/2}(\eta) \\
& +\Big[-\xi_{\mathrm{s}}\,\left(1+{\xi_{\mathrm{s}}}^2\right)^2g_2^4(\xi)\,h_0^4(\xi_{\mathrm{s}}) \\
& +\frac{1}{15}\,\left((5\,{\xi_{\mathrm{s}}}^2+4)\,g_2^4(\xi)+6\right)\Big]C_2^{3/2}(\eta)\Bigg\}
\end{aligned}
\tag{16a}
$$

and

$$\tilde{\omega}_2^{\mathrm{o}} = \frac{6}{5}\left(1+{\xi_{\mathrm{s}}}^2\right)^{3/2}\left(h_0^4(\xi)\,C_0^{3/2}(\eta) - h_2^4(\xi)\,C_2^{3/2}(\eta)\right). \tag{16b}$$

Using the scheme proposed here, the calculation of the higher orders is much lengthier than the calculations just shown, but otherwise identical. For ν_4 for example, the last metric function needed in describing the first post-Newtonian correction, we find

$$\begin{aligned}
{a_0}^2\Delta_2\nu_4^{\mathrm{i}} = &\Bigg[\frac{3}{4}\,\frac{(-1-9\,\eta^4+63\,\eta^4\,\psi^2+6\,\eta^2-54\,\psi^2\,\eta^2+3\,\psi^2)\,\sqrt{1+{\xi_{\mathrm{s}}}^2}\,\mathrm{arccot}(\xi_{\mathrm{s}})}{\eta^2+\psi^2}\\
&-\frac{3}{4}(63\,\psi^2\,{\xi_{\mathrm{s}}}^2\,\eta^4+42\,\eta^4\,\psi^2-6\,\eta^4-9\,{\xi_{\mathrm{s}}}^2\,\eta^4-54\,\eta^2\,\psi^2\,{\xi_{\mathrm{s}}}^2+6\,\eta^2\,{\xi_{\mathrm{s}}}^2\\
&+2\,\eta^2-36\,\psi^2\,\eta^2-{\xi_{\mathrm{s}}}^2+3\,\psi^2\,{\xi_{\mathrm{s}}}^2+2\,\psi^2)\Big/\left((\eta^2+\psi^2)\,\xi_{\mathrm{s}}\sqrt{1+{\xi_{\mathrm{s}}}^2}\right)\Bigg]S_{22}\\
&+\left[-\frac{3}{2}\,\frac{(3\,\eta^2-1)\,\sqrt{1+{\xi_{\mathrm{s}}}^2}\,\mathrm{arccot}(\xi_{\mathrm{s}})}{\eta^2+\psi^2}+\frac{3}{2}\,\frac{2\,\eta^2+3\,\eta^2\,{\xi_{\mathrm{s}}}^2+2\,\psi^2-{\xi_{\mathrm{s}}}^2}{(\eta^2+\psi^2)\,\xi_{\mathrm{s}}\,\sqrt{1+{\xi_{\mathrm{s}}}^2}}\right]S_{02}\\
&-\frac{21}{4}\,\frac{(1+\psi^2)\,(-1+\eta)\,(\eta+1)\,\tilde{\Omega}_1^2}{\xi_{\mathrm{s}}\,\sqrt{1+{\xi_{\mathrm{s}}}^2}}-\frac{9}{2}\,\frac{\gamma_2}{\xi_{\mathrm{s}}\,\sqrt{1+{\xi_{\mathrm{s}}}^2}}\\
&+\frac{45}{16}\,\frac{(1+3\,\psi^2\,\eta^2-\psi^2+\eta^2)\,\mathrm{arccot}(\xi_{\mathrm{s}})}{\xi_{\mathrm{s}}}\\
&-\frac{45}{16}\,\frac{-\psi^2\,{\xi_{\mathrm{s}}}^2+\eta^2\,{\xi_{\mathrm{s}}}^2-{\xi_s}^2+3\,\eta^2\,\psi^2\,{\xi_{\mathrm{s}}}^2+2\,\psi^2\,\eta^2}{{\xi_{\mathrm{s}}}^2\,(1+{\xi_{\mathrm{s}}}^2)}
\end{aligned} \tag{17a}$$

and

$$\begin{aligned}
{a_0}^2\Delta_2\nu_4^{\mathrm{o}} = &\Bigg[\frac{3}{4}\,\frac{(-1-9\,\eta^4+63\,\eta^4\,\psi^2+6\,\eta^2-54\,\psi^2\,\eta^2+3\,\psi^2)\,\sqrt{1+{\xi_{\mathrm{s}}}^2}\,\mathrm{arccot}(\psi)}{\eta^2+\psi^2}\\
&-\frac{3}{4}\psi(27\,\eta^4+96\,\eta^4\,\psi^2+63\,\eta^4\,\psi^4-16\,\eta^2-78\,\psi^2\,\eta^2-54\,\eta^2\,\psi^4\\
&+2\,\psi^2-3+3\,\psi^4)\sqrt{1+{\xi_{\mathrm{s}}}^2}\Big/\left((\eta^2+\psi^2)\,(1+\psi^2)^2\right)\Bigg]S_{22}\\
&+\Bigg[-\frac{3}{2}\,\frac{(3\,\eta^2-1)\,\sqrt{1+{\xi_{\mathrm{s}}}^2}\,\mathrm{arccot}(\psi)}{\eta^2+\psi^2}\\
&+\frac{3}{2}\,\frac{\psi\,(5\,\eta^2+3\,\psi^2\,\eta^2-\psi^2-3)\,\sqrt{1+{\xi_{\mathrm{s}}}^2}}{(\eta^2+\psi^2)\,(1+\psi^2)^2}\Bigg]S_{02}.
\end{aligned} \tag{17b}$$

These source terms are fourth order polynomials in η after having been multiplied with the factor $(\psi^2 + \eta^2)$ from eq. (8). Because of the orthogonality of the Gegenbauer polynomials, ν_4 is thus also a fourth order polynomial in η. This property propagates itself through the successive post-Newtonian orders such that a term ϕ_n is always an n^{th} order polynomial in η. Physically this is because the perturbative-like corrections to the shape of the surface, which are in the form of a *finite* sum of Legendre polynomials (see eq. (5)), give rise to a finite number of multipole moments.

The source terms in eqs (17) contain two of the boundary coefficients S_{k2}, which have to be determined by solving eq. (10), i.e. requiring that the pressure vanish on the boundary. A term containing S_{12} could also be included in eqs (17), but is found to equal zero by applying this boundary condition. Solving eq. (11) and using the abbreviation

$$\beta := \operatorname{arccot}(\xi_{\rm s})$$

one finds

$$\begin{aligned}
S_{22} = &\frac{1}{2}(1+{\xi_{\rm s}}^2)^{(3/2)}\Big(288\,\xi_{\rm s}\,\beta - 45\,\beta^2 + 408\,\beta\,{\xi_{\rm s}}^3 - 54\,\beta^2\,{\xi_{\rm s}}^4 + 1575\,\beta^2\,{\xi_s}^8 \\
&- 378\,\beta^2\,{\xi_{\rm s}}^2 + 1710\,\beta^2\,{\xi_{\rm s}}^6 - 3150\,{\xi_{\rm s}}^7\,\beta - 2370\,{\xi_{\rm s}}^5\,\beta - 179\,{\xi_{\rm s}}^2 \\
&+ 1575\,{\xi_{\rm s}}^6 + 660\,{\xi_{\rm s}}^4\Big)\Big/\Big(3330\,\beta\,{\xi_{\rm s}}^4 - 1965\,{\xi_{\rm s}}^3 + 732\,\beta\,{\xi_{\rm s}}^2 \\
&- 357\,\xi_{\rm s} - 5075\,{\xi_{\rm s}}^5 - 3675\,{\xi_s}^7 + 3675\,\beta\,{\xi_{\rm s}}^8 + 6300\,\beta\,{\xi_{\rm s}}^6 + 27\,\beta\Big).
\end{aligned}$$

The denominator of this expression is proportional to $G_4(\xi_{\rm s})$ of eq. (12) and gives rise to the singularity at ξ_4^*, which has already been discussed. We choose to use the remaining two non-trivial equations extracted from eq. (10) to determine S_{02} and $\tilde{\Omega}_3$. Equivalently, we could have determined γ_4 instead of $\tilde{\Omega}_3$. In either case, the remaining constant can be chosen freely and does not affect the validity of the solution, but instead specifies the Newtonian spheroid of comparison. Because the constants S_{02} and $\tilde{\Omega}_3$ are determined from a linear algebraic system of equations involving S_{22}, they also contain a first order pole at the point ξ_4^*.

The determination of higher orders proceeds identically. One first obtains the Poisson-like equations by expanding eqs (1a–1c) and extracting the coefficients of the desired order in ε. Next one solves these using eq. (6) with ψ replacing ξ, integrating from 0 to $\xi_{\rm s}$ for the interior of the star and from $\xi_{\rm s}$ to ∞ for the exterior. The metric function μ can then be most easily computed by making use of the integral

$$\mu - \lambda = \int_1^\eta (\mu - \lambda)_{,\eta'}\, d\eta', \tag{18}$$

where the endpoint of integration follows from $\lim_{\eta\to 1}(\mu - \lambda) = 0$ (see e.g. eqs (22) and

(23) in [9]). The integrand can be determined from the equations

$$\begin{aligned}\frac{1}{\varrho}(\mu-\lambda)_{,\zeta} &= (\nu+\lambda)_{,\zeta\varrho} + \nu_{,\varrho}\,\nu_{,\zeta} + \lambda_{,\varrho}\,\lambda_{,\zeta} - \mu_{,\zeta}\,(\nu+\lambda)_{,\varrho} \\ &\quad - \mu_{,\varrho}\,(\nu+\lambda)_{,\zeta} - \frac{\tilde{\Omega}^2\varrho^2 e^{2\lambda-2\nu}}{2\,{a_0}^2}\tilde{\omega}_{,\zeta}\tilde{\omega}_{,\varrho}\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{1}{\varrho}(\mu-\lambda)_{,\varrho} &= \frac{1}{2}\left[(\nu+\lambda)_{,\varrho\varrho} - (\nu+\lambda)_{,\zeta\zeta}\right] + \frac{1}{2}\left({\nu_{,\varrho}}^2 + {\lambda_{,\varrho}}^2 - {\nu_{,\zeta}}^2 + {\lambda_{,\zeta}}^2\right) \\ &\quad - \left(\mu_{,\varrho}(\nu+\lambda)_{,\varrho} - \mu_{,\zeta}(\nu+\lambda)_{,\zeta}\right) - \frac{\tilde{\Omega}^2\varrho^2 e^{2\lambda-2\nu}}{4\,{a_0}^2}\left(\tilde{\omega}_{\varrho}^2 - \tilde{\omega}_{\zeta}^2\right)\end{aligned} \tag{20}$$

together with the transformation equation

$$f_{,\eta} = a_0\xi f_{,\zeta} - {a_0}^2\eta(1+\xi^2)\frac{f_{,\varrho}}{\varrho}.$$

Because $\mu-\lambda$ is a polynomial in η to any order of the approximation, integrating eq. (18) is trivial. Finally one uses the boundary condition to determine the boundary coefficients and $\tilde{\Omega}_{i+1}$.

We carried out this procedure for the first four orders of the PN approximation. The results are entirely expressible in terms of elementary functions and in the interior of the star the metric functions are simply polynomials with respect to ψ^2 and η^2. The validity of the results was ensured by confirming that the disc limit of the expansion reduces to that of [10], by showing that the expressions for the gravitational mass and angular momentum found by integrating over the interior of the star are identical to those taken from the far field and by comparing PN-values to those returned by highly accurate numerical calculations.

6.2 Representative Physical Quantites

Physical quantities that are of interest in characterizing a given configuration are its rest mass

$$M_0 = 2\pi Q{a_0}^3 \int_{-1}^{1}\int_{0}^{\xi_\mathrm{s}} \frac{e^{\lambda+2\mu}}{\sqrt{1-\tilde{v}^2}}\left[\left(\psi\xi_\mathrm{B}(\eta)/\xi_\mathrm{s}\right)^2 + \eta^2\right]\,\xi_\mathrm{B}(\eta)/\xi_\mathrm{s}\,d\psi\,d\eta, \tag{21}$$

angular momentum

$$\begin{aligned}J = 2\pi Q{a_0}^4 c \int_{-1}^{1}\int_{0}^{\xi_\mathrm{s}} &\frac{\tilde{\Omega}(1-\gamma)(1-\tilde{\omega})e^{3\lambda-2\nu+2\mu}}{(1-\tilde{v}^2)^{3/2}}[1+(\psi\xi_\mathrm{B}(\eta)/\xi_\mathrm{s})^2](1-\eta^2) \\ &\left[\left(\psi\xi_\mathrm{B}(\eta)/\xi_\mathrm{s}\right)^2 + \eta^2\right]\,\xi_\mathrm{B}(\eta)/\xi_\mathrm{s}\,d\psi\,d\eta,\end{aligned} \tag{22}$$

binding energy

$$
\begin{aligned}
E_b = & \gamma M_0\, c^2 - 2\tilde{\Omega} c/a_0\, J \\
& - 4\pi {a_0}^3 \int_{-1}^{1} \int_{0}^{\xi_{\rm s}} P e^{\nu+\lambda+2\mu} \left[(\psi \xi_{\rm B}(\eta)/\xi_{\rm s})^2 + \eta^2 \right] \xi_{\rm B}(\eta)/\xi_{\rm s}\, d\psi\, d\eta
\end{aligned} \tag{23}
$$

and gravitational mass

$$
M = M_0 - E_b/c^2. \tag{24}
$$

The expression $\left[(\psi \xi_{\rm B}(\eta)/\xi_{\rm s})^2 + \eta^2 \right] \xi_{\rm B}(\eta)/\xi_{\rm s}\, d\psi\, d\eta$ in the integrals comes from applying the coordinate transformation in eq. (5) to the volume element $(\xi^2 + \eta^2)\, d\xi\, d\eta$. As an alternative to eqs (22) and (24), one can choose to calculate the angular momentum and gravitational mass from the far fields of ω and ν respectively and then use eq. (24) in order to find the binding energy. The disadvantage of the far field approach is that $\tilde{\omega}_{i+2}$ must be known in order to find J_i, whereas $\tilde{\omega}_i$ suffices otherwise.

In addition to using the above quantities, we shall characterize configurations by the ratio of polar to equatorial radius

$$
r_{\rm p}/r_{\rm e} = \xi_{\rm B}(\eta=1)/\sqrt{1 + (\xi_{\rm B}(\eta=0))^2},
$$

the polar red shift $Z_{\rm p}$ and "surface potential" V_0

$$
Z_{\rm p} = e^{-\nu(\psi=\xi_{\rm s},\eta=1)} - 1 \equiv e^{-V_0} - 1 = \gamma/(1-\gamma),
$$

the central pressure

$$
P_{\rm c} = P(\psi = 0, \eta = 1)
$$

as well as the angular velocity Ω.

In appendix B, tables providing information about the expansion of these quantities can be found.

6.3 Convergence and Accuracy

As was mentioned in the introduction, we are lucky to have at our disposal a highly accurate numerical code. The AKM code [11, 12] uses a multi-domain spectral method to solve the Einstein equations for perfect fluids in an axially symmetric, stationary spacetime for some specified equation of state. The accuracy reached approaches machine accuracy and has thus been used as a standard [13] to ascertain the accuracy of other numerical codes such as Lorene/rotstar [14], the SF codes [15] or that of KEH [16, 17] (see [13] for further information). Due to the extremely high accuracy, we can use the numerically generated configurations as though they were analytic solutions,

Numerical value	Relative error for different choices of PN-expansion $(i>0)$				
	$\gamma_{i+2}=0$	$M_i=0$	$\tilde{\Omega}_{i+1}=0$	$J_i=0$	$(r_\mathrm{p}/r_\mathrm{e})_i=0$
$e^{V_0}=0.95$	—	7.6×10^{-9}	—	1.4×10^{-8}	—
$\Omega=0.874$	—	—	—	—	2.2×10^{-5}
$M=0.004808\ldots$	3.2×10^{-8}	—	3.6×10^{-5}	8.6×10^{-7}	2.7×10^{-5}
$M_0=0.004936\ldots$	1.0×10^{-7}	4.0×10^{-7}	2.9×10^{-4}	4.5×10^{-7}	4.1×10^{-5}
$P_\mathrm{c}=0.02151\ldots$	1.2×10^{-6}	3.6×10^{-7}	7.7×10^{-4}	1.1×10^{-7}	2.2×10^{-4}
$J=0.00002272\ldots$	1.2×10^{-6}	1.8×10^{-6}	1.4×10^{-3}	—	3.9×10^{-4}
$\frac{r_\mathrm{p}}{r_\mathrm{e}}=0.7659\ldots$	4.3×10^{-7}	3.1×10^{-7}	3.0×10^{-5}	1.8×10^{-7}	—

Table 2: Relative errors for various physical quantities according to the 4^{th} PN order for different choices for the Newtonian spheroids of comparison. The header of columns 2–6 show which equation is satisfied by the respective choice for the constant γ_i, $i>2$. A dash indicates that this quantity was prescribed.

which is what enables us to provide values for the relative errors of physical quantities for example.

In what follows, we shall use units in which $G=c=Q=1$ holds and use the term "Newtonian limit" to refer to the limit in which the Newtonian and relativistic theories agree. In this limit, a_0 goes to zero while ξ_s remains finite, thus resulting in the fact that $M\to0$ and $e^{V_0}\to1$. A "Newtonian" or "Maclaurin" configuration on the other hand is the term we use to refer to the spheroidal figure that one obtains from the Newtonian theory, i.e. from ν_2 together with γ_2 and $\tilde{\Omega}_1$.

In Table 2 a comparison was made of different choices for the Newtonian spheroid of comparison. This amounts to different choices for the constants γ_i, $i>2$. One can see in this example of a configuration near the Newtonian limit, that the choice made can lead to differences of a few orders of magnitude for relative errors. This surprising result can be seen, moreover, to hold over a large range of values for the parameter e^{V_0} in Figure 1. On the left hand side of this figure, the PN approximation with $\tilde{\Omega}_i=0$, $i>1$ is depicted. The various orders react as they must in the vicinity of the Newtonian limit $1-e^{V_0}=0$: each new order brings about a noticeable improvement. As one moves away from this limit however, the curves cross each other and it turns out that higher orders render a worse approximation than lower ones. The right hand side of the figure tells a very different story. Here the PN approximation with $\gamma_{i+1}=0$, $i>1$ is depicted. Each additional term in the PN approximation brings about a marked improvement in accuracy and, moreover, the relative error is more than an order of magnitude lower than on the left hand side.

Imagine for a moment that one had calculated the PN approximation presented here without being in possession of numerical values. Furthermore, let us imagine that one

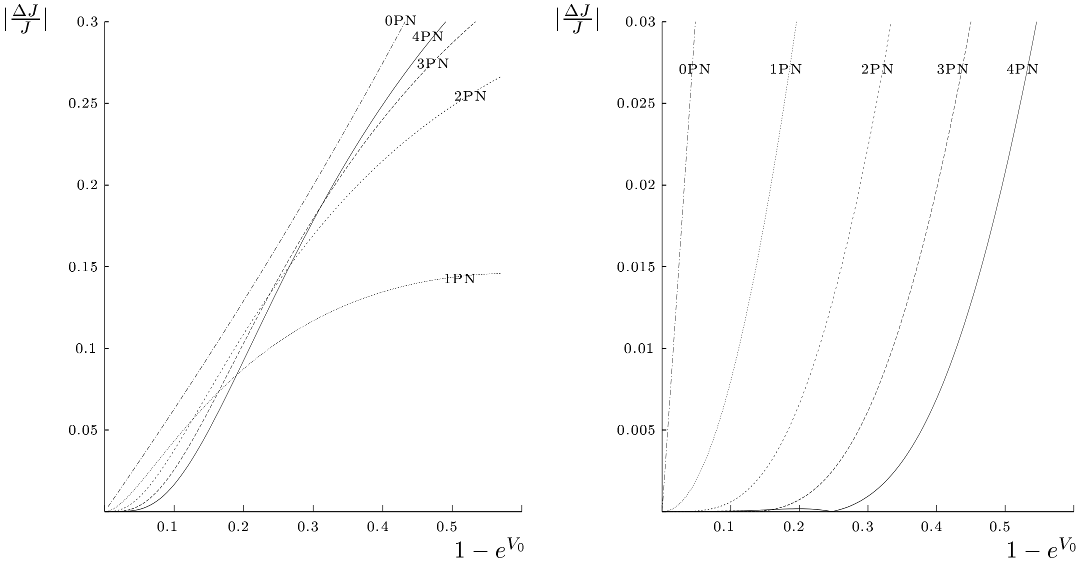

Figure 1: The relative error of J versus $1-e^{V_0}$ for configurations with $\Omega = 0.874$. On the left the PN-expansion with $\tilde{\Omega}_i = 0$, $i > 1$, was used and on the right with $\gamma_{i+1} = 0$. Here 0PN refers to the Maclaurin solution, 1PN to the first order of the PN approximation etc.

had decided from the outset to prescribe $\tilde{\Omega}_i = 0$, $i > 1$. Then one would have been able to produce the plot on the left hand side of Figure 2 without the numerical curve. It would have been natural to suppose that the PN series converges toward the correct solution and that the fourth order of the PN approximation almost provides the correct value for J even up to values for M of 0.12. Had one chosen $\gamma_{i+1} = 0$ instead, then one would have produced the right hand side of Figure 2 without the numerical curve and come to the same conclusions regarding the convergence of the PN approximation. In that case, however, one would have been correct.

Although curves depicting relative errors of other physical quantities may look quite different from those for J shown in Figure 1, they also have many important aspects in common. The choice $\gamma_{i+1} = 0$, $i > 1$, leads to much smaller relative errors than for $\tilde{\Omega}_i = 0$ and one tends to find improvement with increasing order even far away from the Newtonian limit. These properties hold for a wide range of Ω values and the relative errors tend to decrease with decreasing angular velocity so long as one does not come too close to a singularity in the parameter space.

One well known technique for improving on the PN approximation is the use of the Padé approximant, which approximates a truncated series by a quotient of two polynomials and is discussed with reference to the PN approximation in [18]. For the disc limit of the solution considered here, it has been shown in [10] that the Padé approximant provides a far better approximation of the analytic solution given in [19, 20]

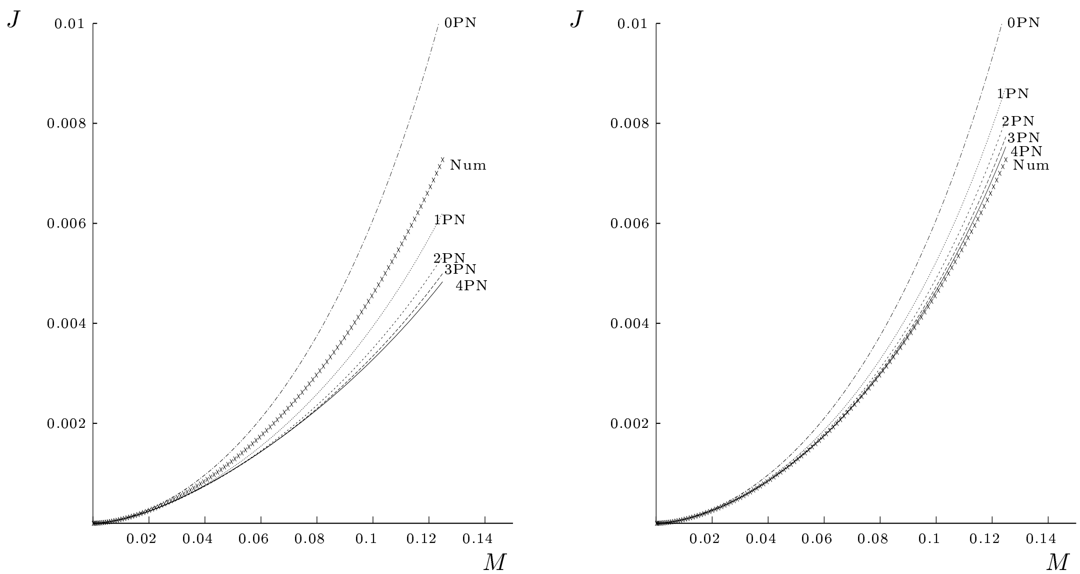

Figure 2: A plot of J over M for configurations with $\Omega = 0.874$. On the left the PN-expansion with $\tilde{\Omega}_i = 0$, $i > 1$, was used and on the right with $\gamma_{i+1} = 0$. Num refers to the numerical solution, 0PN to the Maclaurin solution, 1PN to the first order of the PN approximation etc.

than the PN approximation itself. In the case of the Maclaurin spheroids this turns out to be true as well, especially for $\Omega < 0.8$. We see in Table 3 how well the Padé approximant with a polynomial of sixth order in the numerator and second order in the denominator converges to the correct solution. Most likely this technique would be even more effective when applied to a somewhat higher order of the approximation. Even up to the fourth order, the PN approximation turns out to be roughly comparable to older numerical codes even for highly relativistic configurations. An impressive illustration of its applicability in such highly relativistic regimes can be found in Fig. 3. In this figure, the meridional cross section of a configuration with a central pressure of 1 and a radius ratio of 0.7 is depicted. One can see that the surface predicted by the fourth order PN approximation is almost indistinguishable from the numerical values.

For a more detailed comparison with numerical values and a more complete account regarding the derivation of the iterative scheme and the singularities in parameter space, the reader is referred to [8]

7 Conclusion

In this paper, an iterative procedure to enable the explicit calculation of any order of the PN approximation of the Maclaurin spheroids was devised. This was made possible

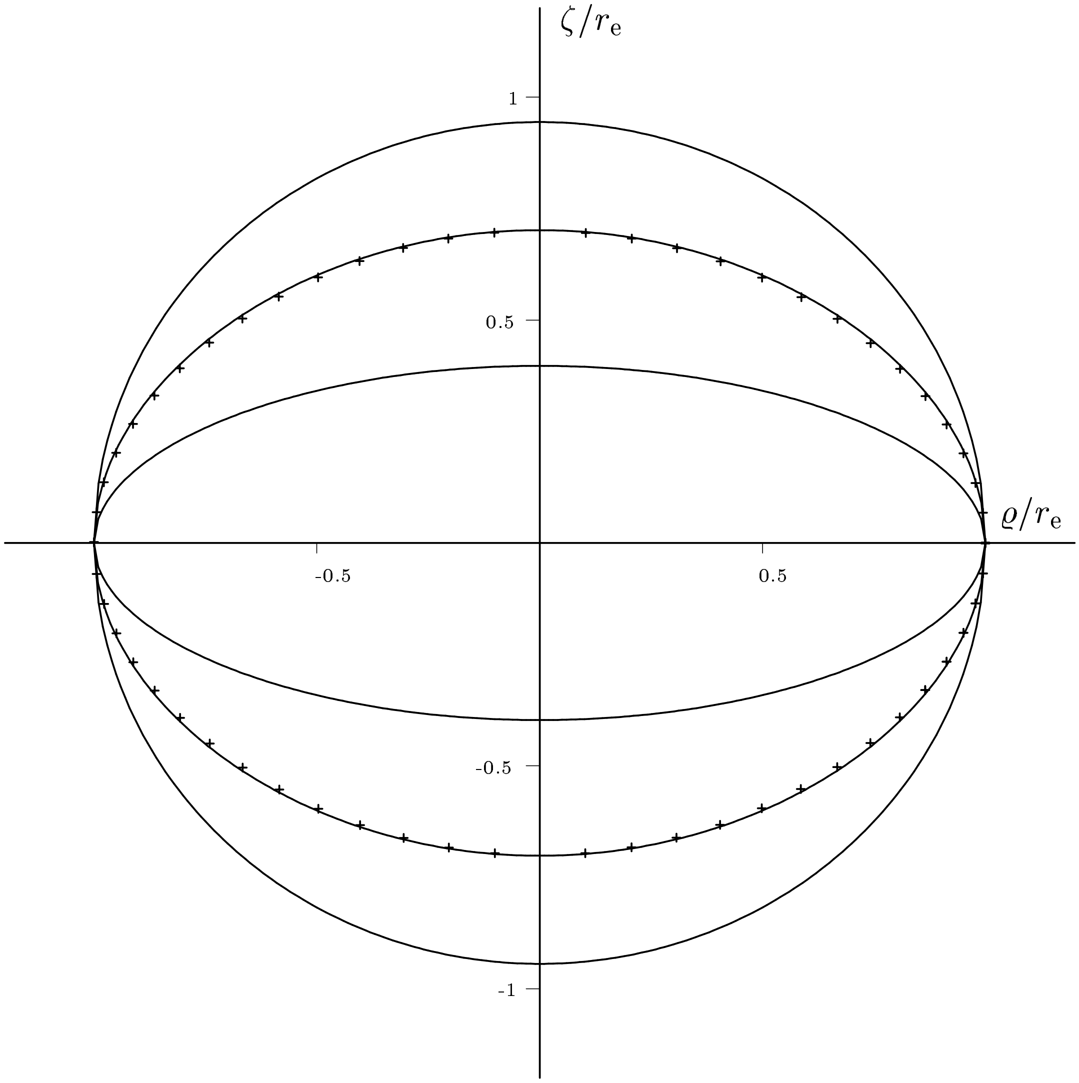

Figure 3: The meridional cross-section of a highly relativistic configuration. For fixed values of ξ_{s} and a_0, the outermost curve represents the Maclaurin spheroid, the innermost was calculated using the first order approximation and the curve between these using the fourth order approximation. The crosses are numerical values.

	0PN	1PN	2PN	3PN	4PN	Padé	AKM
e^{V_0}	0.7	0.7	0.7	0.7	0.7	0.7	0.7
Ω	0.3	0.3	0.3	0.3	0.3	0.3	0.3
M ($\times 10^{-2}$)	7.94	6.17	6.244	6.2490	6.25013	6.25055	6.25070
M_0 ($\times 10^{-2}$)	7.94	7.60	7.589	7.5864	7.58590	7.58554	7.58553
P_c ($\times 10^{-1}$)	1.47	2.14	2.434	2.5627	2.61841	2.66099	2.66064
J ($\times 10^{-4}$)	6.90	6.34	6.141	6.0744	6.05313	6.04321	6.04352
$\frac{r_p}{r_e}$ ($\times 10^{-1}$)	9.73	9.79	9.801	9.8041	9.80538	9.80606	9.80628

Table 3: Values of various physical quantities according to different orders of the PN approximation. 0PN stands for the Maclaurin solution, 1PN for the first PN approximation etc. The PN approximation with $\gamma_i = 0$, $i > 2$ was used and the Padé approximant was applied to the fourth order solution.

by introducing coordinates tailored to the unknown surface of the star, by requiring that this surface's representation be a terminating sum and by realizing that eq. (6) can be used in the new coordinates without alteration. The PN expansion was carried out explicitly to the fourth order and the resulting expressions contained only elementary functions.

It was proved that the n^{th} PN approximation has a first order pole at ξ^*_{2n+2}, the onset of the n^{th} axisymmetric, harmonic mode of secular instability. The radius of convergence of the series becomes zero at these points, thereby dividing the ξ_s-ε parameter space into rectangles with "impermeable" walls that accumulate about (but not at) the line $\xi_s = 0$. Since the PN approximation appears to converge even in the highly relativistic regime, it seems likely that no quasi-stationary, axisymmetric sequence of solutions leads from an extended, three dimensional configuration to the disc limit – all such configurations would have to pass through an infinite number of such impermeable walls.

The convergence of the PN approximation was shown to depend strongly on the choice of the Newtonian configuration of comparison. A poor choice can render the approximation useless in the relativistic regime, but a good one was shown to converge quite well, especially when aided by the Padé approximant. These results can be taken as a word of warning, reminding the researcher that the PN approximation can be very sensitive to alterations that may have no direct physical consequences. On the other hand, they also demonstrate that in the best of circumstances, the PN expansion can yield a very good approximation to highly relativistic configurations, well beyond its guaranteed region of validity.

A The functions $g_l^m(\psi)$ and $h_l^m(\psi)$

$$
\begin{aligned}
g_0^2 &= 1\\
g_2^2 &= -\frac{3\,\psi^2}{2} - \frac{1}{2}\\
g_4^2 &= \frac{35}{8}\,\psi^4 + \frac{15}{4}\,\psi^2 + \frac{3}{8}\\
g_6^2 &= -\frac{231}{16}\,\psi^6 - \frac{315}{16}\,\psi^4 - \frac{105}{16}\,\psi^2 - \frac{5}{16}
\end{aligned}
$$

$$
\begin{aligned}
h_0^2 &= \operatorname{arccot}(\psi)\\
h_2^2 &= \frac{3\,\psi}{2} + \left(-\frac{3\,\psi^2}{2} - \frac{1}{2}\right)\operatorname{arccot}(\psi)\\
h_4^2 &= -\frac{35\,\psi^3}{8} - \frac{55\,\psi}{24} + \left(\frac{35}{8}\,\psi^4 + \frac{15}{4}\,\psi^2 + \frac{3}{8}\right)\operatorname{arccot}(\psi)\\
h_6^2 &= \frac{231\,\psi^5}{16} + \frac{119\,\psi^3}{8} + \frac{231\,\psi}{80}\\
&\quad + \left(-\frac{231}{16}\,\psi^6 - \frac{315}{16}\,\psi^4 - \frac{105}{16}\,\psi^2 - \frac{5}{16}\right)\operatorname{arccot}(\psi)
\end{aligned}
$$

$$
\begin{aligned}
g_0^3 &= 1\\
g_2^3 &= -4\,\psi^2 - 1\\
g_4^3 &= 16\,\psi^4 + 12\,\psi^2 + 1\\
g_6^3 &= -64\,\psi^6 - 80\,\psi^4 - 24\,\psi^2 - 1
\end{aligned}
$$

$$
\begin{aligned}
h_0^3 &= 1 - \frac{\psi}{\sqrt{1+\psi^2}}\\
h_2^3 &= -\frac{4\,\psi^2}{3} - \frac{1}{3} + \frac{\frac{4}{3}\,\psi^3 + \psi}{\sqrt{\psi^2+1}}\\
h_4^3 &= \frac{16\,\psi^4}{5} + \frac{12\,\psi^2}{5} + \frac{1}{5} + \frac{-\frac{16}{5}\,\psi^5 - 4\,\psi^3 - \psi}{\sqrt{\psi^2+1}}\\
h_6^3 &= -\frac{64\,\psi^6}{7} - \frac{80\,\psi^4}{7} - \frac{24\,\psi^2}{7} - \frac{1}{7} + \frac{\frac{64}{7}\,\psi^7 + 16\,\psi^5 + 8\,\psi^3 + \psi}{\sqrt{\psi^2+1}}
\end{aligned}
$$

$$
\begin{aligned}
g_0^4 &= 1\\
g_2^4 &= -\frac{15\,\psi^2}{2} - \frac{3}{2}\\
g_4^4 &= \frac{315}{8}\,\psi^4 + \frac{105}{4}\,\psi^2 + \frac{15}{8}\\
g_6^4 &= -\frac{3003}{16}\,\psi^6 - \frac{3465}{16}\,\psi^4 - \frac{945}{16}\,\psi^2 - \frac{35}{16}
\end{aligned}
$$

$$
\begin{aligned}
h_0^4 &= \frac{1}{2}\,\mathrm{arccot}(\psi) - \frac{\psi}{2\,(\psi^2+1)}\\
h_2^4 &= \left(-\frac{5\,\psi^2}{8} - \frac{1}{8}\right)\mathrm{arccot}(\psi) + \frac{(15\,\psi^2+13)\,\psi}{24\,(\psi^2+1)}\\
h_4^4 &= \left(\frac{21}{16}\,\psi^4 + \frac{7}{8}\,\psi^2 + \frac{1}{16}\right)\mathrm{arccot}(\psi) - \frac{(315\,\psi^4+420\,\psi^2+113)\,\psi}{240\,(\psi^2+1)}\\
h_6^4 &= \left(-\frac{429}{128}\,\psi^6 - \frac{495}{128}\,\psi^4 - \frac{135}{128}\,\psi^2 - \frac{5}{128}\right)\mathrm{arccot}(\psi)\\
&\quad + \frac{(15015\,\psi^6+27335\,\psi^4+14273\,\psi^2+1873)\,\psi}{4480\,(\psi^2+1)}
\end{aligned}
$$

B Tables of Various Physical Quantities

This appendix contains tables with the numerical values for the post-Newtonian coefficients of the quantities introduced in section 6.2 for various values of ξ_s. In all the tables, we have chosen $\gamma_i = 0$, $i > 2$, whence we find

$$
\gamma = \gamma_2 \varepsilon^2 = \frac{3}{4}\sqrt{1+{\xi_\mathrm{s}}^2}\left(\mathrm{arccot}(\xi_\mathrm{s})(1+{\xi_\mathrm{s}}^2) - \xi_\mathrm{s}\right)\varepsilon^2.
$$

Taking into account

$$
{a_0}^2 = \frac{3c^2}{8\pi\,Q\xi_\mathrm{s}\sqrt{1+\xi_\mathrm{s}^2}}\,\varepsilon^2,
$$

we would find, for example, the following values for the second PN approximation of configuration with $\xi_\mathrm{s} = 0.5$, $\varepsilon = 0.7$:

$$
\begin{aligned}
a_0 &\approx\; 0.32346\\
e^{V_0} &=\; 1-\gamma \approx\; 0.63681\\
\Omega = \tilde{\Omega}/a_0 &\approx \frac{1}{a_0}\left(0.54175\,\varepsilon + 0.26458\,\varepsilon^3 - 0.32261\,\varepsilon^5\right) \approx\; 1.285
\end{aligned}
$$

$$
\begin{aligned}
M &\approx \left(2.6180 - 1.3693\,\varepsilon^2 + 1.7071\,\varepsilon^4\right)\ {a_0}^3 \approx\ 0.07976 \\
M_0 &\approx \left(2.6180 - 0.58922\,\varepsilon^2 + 0.99562\,\varepsilon^4\right)\ {a_0}^3 \approx\ 0.08692 \\
P_\mathrm{c} &\approx \left(0.87655 - 0.29553\,\varepsilon^2 - 0.21693\,\varepsilon^4\right) {a_0}^2 \approx\ 0.07111 \\
J &\approx \left(1.5346 + 1.0605\,\varepsilon^2 + 0.24311\,\varepsilon^4\right)\ {a_0}^5 \approx\ 0.007480 \\
\frac{r_\mathrm{p}}{r_\mathrm{e}} &\approx \left(0.44721 - 0.55974\,\varepsilon^2 - 0.23039\,\varepsilon^4\right) \approx\ 0.2283.
\end{aligned}
$$

Acknowledgments

Many thanks to R. Meinel, M. Ansorg and J. Ehlers for the valuable discussions and helpful advice.

References

[1] S. Chandrasekhar. The Post-Newtonian Effects of General Relativity on the Equilibrium of Uniformly Rotating Bodies II. The Deformed Figures of the Maclaurin Spheroids. *Astroph. J.*, **147**, 334 (1967).

[2] J.M. Bardeen. A Reexamination of the Post-Netwonian Maclaurin Spheroids. *Astroph. J.*, **167**, 425 (1971).

[3] J.B. Hartle & D.H. Sharp. Variational Principle for the Equilibrium of a Relativistic, Rotating Star. *Astrophys. J.*, **147**, 317 (1967).

[4] L. Lichtenstein. *Gleichgewichtsfiguren Rotierender Flüssigkeiten* (Springer, Berlin, 1933).

[5] L. Lindblom. On the Symmetries of Equilibrium Stellar Models. *Philos. Trans. R. Soc. London, Ser. A*, **340**, 353 (1992).

[6] I. Hachisu & Y. Eriguchi. Bifurcation Points on the Maclaurin Sequence. *Publ. Astron. Soc. Japan*, **36**, 497 (1984).

[7] M. Ansorg, A. Kleinwächter & R. Meinel. Uniformly Rotating Axisymmetric Fluid Configurations Bifurcating from Highly Flattened Maclaurin Spheroids. *Mon. Not. R. Astron. Soc.*, **339**, 515 (2003).

[8] D. Petroff. *Die Maclaurin-Ellipsoide in post-Newtonscher Näherung beliebig hoher Ordnung*. Ph.D. thesis, Friedrich-Schiller-Universität, Jena (2003).

[9] R. Meinel. Black Holes: A Physical Route to the Kerr Metric. *Annalen Phys.*, **11**, 509 (2002).

	Coefficients of ε^i for the dimensionless angular velocity $\tilde{\Omega}$				
ξ_s	ε^1	ε^3	ε^5	ε^7	ε^9
0.00	1.0854019	-0.95903173	-0.21316642	-0.09051296	-0.04918981
0.01	1.0716302	-0.92995651	-0.19734833	-0.11039829	-0.03104705
0.02	1.0579561	-0.90160729	-0.18278329	-0.12768605	-0.01297664
0.03	1.0443818	-0.8741343	-0.16911244	-0.14231975	0.00475476
0.04	1.0309094	-0.84772939	-0.15613789	-0.15411481	0.02219393
0.05	1.0175410	-0.82264177	-0.14417187	-0.16467618	0.01858310
0.06	1.0042785	-0.7992021	-0.13496336	-0.19414756	-1.0164212
0.07	0.99112387	-0.77786047	-0.13435142	-0.44265195	24.323369
0.08	0.97807889	-0.75924895	-0.16091713	-5.9366901	413.17220
0.09	0.96514535	-0.74428879	-0.28117178	12.104546	-22.734741
0.10	0.95232494	-0.73438506	-0.8296993	31.560268	-462.39599
0.11	0.93961928	-0.73180435	-8.5365954	809.03790	-58587.792
0.12	0.92702988	-0.74047382	5.2676596	10.231313	2190.3915
0.13	0.91455822	-0.76787434	4.9354647	-55.887234	1224.6781
0.14	0.90220565	-0.83029178	7.3381300	-134.37904	3062.2576
0.15	0.88997346	-0.97129085	14.854009	-437.85365	15041.748
0.16	0.87786287	-1.3603363	49.619624	-3139.3425	234545.37
0.17	0.865875	-3.9363526	1171.2187	-606631.48	3.87526090×10^8
0.18	0.85401088	1.9317681	-11.780334	2884.4373	-326822.46
0.19	0.84227147	0.57651868	13.454904	127.48863	100.77384
0.20	0.83065764	0.27773007	6.1769373	69.330710	524.99073
0.30	0.72155838	0.09314481	-0.2608214	1.6431116	5.7294717
0.40	0.62537562	0.1993343	-0.43302451	0.15006752	1.3582894
0.50	0.54174791	0.26458306	-0.32260971	-0.27140215	0.58262038
0.60	0.46979598	0.28819495	-0.17835186	-0.34929691	0.09017830
0.70	0.40833633	0.28520112	-0.06646413	-0.28550798	-0.13370899
0.80	0.35606582	0.26835131	0.00483724	-0.19391875	-0.17918978
0.90	0.3116946	0.24576778	0.04504743	-0.11763146	-0.15297576
1.00	0.27402709	0.22194776	0.06517545	-0.06422083	-0.11177801
1.10	0.2420016	0.19912131	0.07339547	-0.02967889	-0.07575039
1.20	0.21470198	0.17824581	0.07493751	-0.00829155	-0.04899455
1.30	0.19135249	0.15961339	0.07292802	0.00454162	-0.0304128
1.40	0.17130433	0.14319157	0.06916914	0.01198574	-0.01789323
1.50	0.15401892	0.1288049	0.06467418	0.01608587	-0.00957769
1.60	0.13905099	0.11622792	0.05999884	0.01812893	-0.00409702
1.70	0.12603306	0.10523057	0.05543697	0.01891680	-0.00050821
1.80	0.11466173	0.09559856	0.05113403	0.01894809	0.00182048
1.90	0.10468625	0.08714120	0.04715237	0.01853317	0.00330817
2.00	0.09589882	0.07969281	0.04350861	0.0178655	0.00423273

Table 4: Expansion coefficients of $\tilde{\Omega}$ for various values of ξ_s with $\gamma_i = 0$, $i > 2$.

	Coefficients of ε^i for the gravitational mass M				
ξ_s	ε^0	ε^2	ε^4	ε^6	ε^8
0.00	0	0	0	0	0
0.01	0.04189209	0.04246214	0.04840595	0.05310172	0.06138033
0.02	0.08380931	0.08115227	0.09502374	0.09931087	0.1165261
0.03	0.1257768	0.11627705	0.14034491	0.13816091	0.16779538
0.04	0.16781969	0.14806521	0.18487986	0.1690188	0.2181801
0.05	0.20996311	0.17677919	0.22928743	0.19157084	0.27677559
0.06	0.25223219	0.2027326	0.27473656	0.21060612	0.5979699
0.07	0.29465207	0.22631723	0.32398279	0.28537561	-6.0060957
0.08	0.33724788	0.24804689	0.38498653	2.0152694	-124.06068
0.09	0.38004475	0.26863219	0.48597184	-3.9985889	-0.15355203
0.10	0.42306781	0.28911548	0.77113254	-11.621492	155.23679
0.11	0.4663422	0.31113215	4.1859186	-339.22981	24417.212
0.12	0.50989305	0.33746264	-1.8588230	-10.933727	-1022.9939
0.13	0.5537455	0.37333934	-1.7708758	21.555107	-531.17370
0.14	0.59792467	0.43007081	-3.0839441	61.463611	-1420.9588
0.15	0.6424557	0.5377904	-7.4131614	226.89235	-7777.7392
0.16	0.68736372	0.81351169	-28.767430	1830.3581	-136275.58
0.17	0.73267386	2.5952925	-770.75767	398608.22	-2.54467540 $\times 10^8$
0.18	0.77841126	-1.4529533	11.397074	-2110.8820	244860.84
0.19	0.82460105	-0.51954369	-10.093076	-89.164129	226.04238
0.20	0.87126836	-0.3180264	-4.8968261	-57.161748	-380.38457
0.30	1.3697344	-0.37666528	0.81904278	-2.7961160	-9.8763842
0.40	1.9435987	-0.82904344	1.5102380	-0.6053959	-3.4289311
0.50	2.6179939	-1.3693167	1.7070560	0.53885692	-2.1926160
0.60	3.4180528	-1.9472924	1.6481885	1.2005744	-0.98288389
0.70	4.3689082	-2.5527128	1.4829837	1.4507518	-0.00014983
0.80	5.4956927	-3.1933102	1.3002647	1.4514458	0.57039853
0.90	6.8235392	-3.8834294	1.1398464	1.3414215	0.81909493
1.00	8.3775804	-4.6386450	1.0140861	1.1999932	0.88499186
1.10	10.182949	-5.4737714	0.92337032	1.0630189	0.86508412
1.20	12.264778	-6.4024064	0.86407489	0.94327903	0.81216639
1.30	14.648199	-7.4370306	0.83193824	0.84322071	0.75142342
1.40	17.358347	-8.5892485	0.82324479	0.76132333	0.69360857
1.50	20.420352	-9.8700213	0.83510223	0.69491426	0.64259313
1.60	23.859349	-11.289852	0.86539261	0.64129035	0.59918663
1.70	27.700470	-12.858921	0.91263339	0.59809715	0.56295176
1.80	31.968847	-14.587185	0.97583498	0.56340574	0.53302851
1.90	36.689613	-16.484440	1.0543810	0.53567923	0.50848981
2.00	41.887902	-18.560374	1.1479349	0.51370787	0.48848149

Table 5: Expansion coefficients of $\frac{M}{{a_0}^3 Q}$ for various values of ξ_s with $\gamma_i = 0$, $i > 2$.

	Coefficients of ε^i for the rest mass M_0				
ξ_s	ε^0	ε^2	ε^4	ε^6	ε^8
0.00	0	0	0	0	0
0.01	0.04189209	0.05245596	0.06452339	0.07673604	0.09064350
0.02	0.08380931	0.10138291	0.12467777	0.14552079	0.17045889
0.03	0.1257768	0.14697999	0.1810503	0.20649775	0.24117379
0.04	0.16781969	0.18946928	0.23424172	0.25967447	0.30478958
0.05	0.20996311	0.22910739	0.28499562	0.30546133	0.36899426
0.06	0.25223219	0.26620288	0.33455944	0.34955489	0.68695096
0.07	0.29465207	0.30114322	0.38575957	0.4525081	-5.9229587
0.08	0.33724788	0.3344386	0.44661698	2.2159335	-123.76573
0.09	0.38004475	0.36679666	0.5454014	-3.7541975	-0.6779804
0.10	0.42306781	0.39925742	0.82632859	-11.306658	154.00435
0.11	0.4663422	0.43345453	4.2348294	-338.55795	24389.392
0.12	0.50989305	0.47216724	-1.8183462	-10.679058	-1027.1961
0.13	0.5537455	0.52062728	-1.7412446	21.991097	-535.19617
0.14	0.59792467	0.59014308	-3.0682419	62.144578	-1430.4764
0.15	0.6424557	0.71084828	-7.4164107	228.15983	-7811.5324
0.16	0.68736372	0.99975724	-28.802435	1834.1087	-136523.76
0.17	0.73267386	2.7949289	-770.92714	398680.63	-2.54506740×10^8
0.18	0.77841126	-1.2397211	11.483888	-2103.2027	244823.57
0.19	0.82460105	-0.29250893	-10.086248	-86.857421	270.08015
0.20	0.87126836	-0.07698001	-4.9219300	-55.908647	-364.07926
0.30	1.3697344	0.01672239	0.56758844	-2.1242466	-9.5006993
0.40	1.9435987	-0.25789658	1.0217665	0.07622924	-2.9245661
0.50	2.6179939	-0.58921661	0.99559791	1.0782019	-1.3199037
0.60	3.4180528	-0.92033156	0.74161054	1.5062120	0.04353243
0.70	4.3689082	-1.2337985	0.40781773	1.5054690	0.95554101
0.80	5.4956927	-1.5299374	0.07578544	1.2771041	1.3384899
0.90	6.8235392	-1.8155767	-0.22248284	0.97101036	1.3676277
1.00	8.3775804	-2.0987311	-0.48106395	0.66449818	1.2237982
1.10	10.182949	-2.3866409	-0.70436865	0.38738975	1.0182639
1.20	12.264778	-2.6853269	-0.89947761	0.14629252	0.80533928
1.30	14.648199	-2.9996963	-1.0731453	-0.06144076	0.60709587
1.40	17.358347	-3.3337847	-1.2309272	-0.24120271	0.43012826
1.50	20.420352	-3.6909893	-1.3771251	-0.39850746	0.27444199
1.60	23.859349	-4.0742533	-1.5149872	-0.53811594	0.13764981
1.70	27.700470	-4.4862014	-1.6469456	-0.66387001	0.01679751
1.80	31.968847	-4.9292361	-1.7748240	-0.7787784	-0.09091482
1.90	36.689613	-5.4056047	-1.8999983	-0.88516836	-0.18789822
2.00	41.887902	-5.9174456	-2.0235176	-0.98483424	-0.27614163

Table 6: Expansion coefficients of $\frac{M_0}{{a_0}^3 Q}$ for various values of ξ_s with $\gamma_i = 0$, $i > 2$.

	Coefficients of ε^i for the central pressure $P_{\rm c}$				
$\xi_{\rm s}$	ε^0	ε^2	ε^4	ε^6	ε^8
0.00	0	0	0	0	0
0.01	0.00061857	-0.00172000	0.00162459	-0.00077003	0.00044127
0.02	0.00243630	-0.00686564	0.00710374	-0.00491391	0.00493335
0.03	0.00539833	-0.01545281	0.01766303	-0.01697127	0.02537089
0.04	0.00945262	-0.02755907	0.03514794	-0.04585596	0.09808811
0.05	0.01454978	-0.04334499	0.06242121	-0.11038522	0.34561893
0.06	0.02064301	-0.06308734	0.10409423	-0.25635755	1.3551282
0.07	0.02768796	-0.08723187	0.16798563	-0.63884234	4.1589896
0.08	0.03564262	-0.11648064	0.26842379	-3.0575859	120.34604
0.09	0.04446728	-0.15194212	0.43563224	0.41855319	20.090197
0.10	0.05412437	-0.19540392	0.75876395	-2.5871083	4.3812886
0.11	0.06457839	-0.24986232	2.3812791	-100.52138	7005.1522
0.12	0.07579584	-0.32064223	0.88116263	-29.802523	-234.29858
0.13	0.08774509	-0.41805092	2.3107894	-34.315938	388.81769
0.14	0.10039634	-0.5647382	5.0552036	-87.445248	1669.9741
0.15	0.11372151	-0.82158864	12.978500	-348.51779	11069.532
0.16	0.12769416	-1.4259431	52.306181	-3108.6680	224222.47
0.17	0.14228943	-5.0951044	1490.6987	-762280.11	4.84805780 $\times 10^8$
0.18	0.15748397	3.0709510	-44.367825	4384.7277	-570869.92
0.19	0.17325585	1.1138325	19.748422	63.519803	-3223.0377
0.20	0.18958447	0.63815819	11.158362	84.961966	91.304718
0.30	0.3798125	-0.11854446	1.3921945	5.7408680	18.408063
0.40	0.61093217	-0.28101446	0.41114756	2.7524237	4.7395486
0.50	0.8765547	-0.29553405	-0.21692694	1.7597087	3.1658578
0.60	1.1744310	-0.19762251	-0.64456449	0.84355303	2.7155097
0.70	1.5044260	-0.01447731	-0.86930239	0.02192268	1.9510756
0.80	1.8673640	0.23179265	-0.92120251	-0.58277995	1.0264413
0.90	2.2644107	0.52592119	-0.84400036	-0.95208863	0.2108996
1.00	2.6967662	0.85851446	-0.67570587	-1.1260297	-0.39050992
1.10	3.1655292	1.2242483	-0.44338108	-1.1551896	-0.78090453
1.20	3.6716491	1.6203250	-0.16452438	-1.0815067	-1.0002778
1.30	4.2159215	2.0454170	0.15012064	-0.93498189	-1.0911717
1.40	4.7990015	2.4990135	0.49416461	-0.73571739	-1.0879378
1.50	5.4214241	2.9810379	0.86390299	-0.49678082	-1.0156282
1.60	6.0836241	3.4916304	1.2572490	-0.22655363	-0.89172864
1.70	6.7859549	4.0310276	1.6730773	0.06961668	-0.72822437
1.80	7.5287040	4.5994981	2.1108281	0.38832489	-0.53328772
1.90	8.3121058	5.1973091	2.5702700	0.72740591	-0.31249925
2.00	9.1363524	5.8247100	3.0513580	1.0854878	-0.06968863

Table 7: Expansion coefficients of $\frac{P_{\rm c}}{{a_0}^2 Q^2}$ for various values of $\xi_{\rm s}$ with $\gamma_i = 0$, $i > 2$.

	Coefficients of ε^i for angular momentum J				
ξ_s	ε^0	ε^2	ε^4	ε^6	ε^8
0.00	0	0	0	0	0
0.01	0.00519817	0.01064142	0.01709874	0.02545363	0.03580801
0.02	0.01452484	0.02955774	0.04621122	0.06832484	0.09446591
0.03	0.02637109	0.05341626	0.08104960	0.11945215	0.16152228
0.04	0.04014030	0.08105972	0.11900741	0.17572839	0.23030652
0.05	0.05548233	0.11191984	0.15848091	0.23645053	0.2981537
0.06	0.07216065	0.14575859	0.19860914	0.30636975	0.52287242
0.07	0.09000348	0.18260101	0.23993267	0.43348251	-4.1934547
0.08	0.10888064	0.22274743	0.28745884	1.8835269	-96.001495
0.09	0.12869081	0.26685311	0.36460134	-2.9069282	-1.4043833
0.10	0.14935376	0.3161068	0.60538199	-9.8156498	137.90767
0.11	0.17080538	0.37259919	3.8855222	-326.89627	23595.306
0.12	0.19299422	0.44011835	-2.1803366	-9.1109666	-1078.5552
0.13	0.21587897	0.52604717	-2.3712087	28.000940	-642.86950
0.14	0.23942672	0.64663729	-4.3174390	82.390748	-1873.3462
0.15	0.26361148	0.84557362	-10.556837	317.16019	-10852.688
0.16	0.28841317	1.2930930	-41.568836	2647.8340	-197589.40
0.17	0.31381674	3.9378673	-1128.4291	585246.47	-3.73910890 $\times 10^8$
0.18	0.33981149	-1.9051257	7.2345391	-2888.5244	340738.03
0.19	0.36639046	-0.4890684	-17.726915	-178.76932	-44.181794
0.20	0.39355005	-0.13577661	-9.2222335	-105.05367	-807.13712
0.30	0.69802485	0.52311868	-0.57132492	-4.7290425	-21.880756
0.40	1.0714436	0.79662582	0.05656033	-1.1626724	-5.0786338
0.50	1.5346441	1.0605245	0.24310865	-0.220009	-1.5315446
0.60	2.1149817	1.3709717	0.30021959	0.05678398	-0.10285
0.70	2.8447368	1.7637678	0.33048771	0.03348169	0.45890186
0.80	3.7607272	2.2682180	0.38836636	-0.11923423	0.53433897
0.90	4.9043081	2.9119034	0.50500367	-0.30541288	0.35919637
1.00	6.3214817	3.7232237	0.70019629	-0.47535233	0.07990005
1.10	8.0630209	4.7327036	0.98941174	-0.60511924	-0.22477905
1.20	10.184575	5.9736140	1.3874168	-0.68287074	-0.51722445
1.30	12.746756	7.4822472	1.9099198	-0.70137912	-0.78102489
1.40	15.815199	9.2980323	2.5742218	-0.65447365	-1.0093842
1.50	19.460609	11.463577	3.3994232	-0.53552549	-1.1992419
1.60	23.758798	14.024683	4.4064527	-0.33688721	-1.3485367
1.70	28.790709	17.030343	5.6180320	-0.04974338	-1.4550308
1.80	34.642430	20.532741	7.0586270	0.33587825	-1.5158600
1.90	41.405215	24.587246	8.7544002	0.8310431	-1.5274045
2.00	49.175489	29.252409	10.733173	1.4478377	-1.4852922

Table 8: Expansion coefficients of $\frac{J}{{a_0}^5\,Q^{3/2}}$ for various values of ξ_s with $\gamma_i = 0$, $i > 2$.

	Coefficients of ε^i for the radius ratio $r_{\mathrm{p}}/r_{\mathrm{e}}$				
ξ_{s}	ε^0	ε^2	ε^4	ε^6	ε^8
0.00	0	0	0	0	0
0.01	0.0099995	-0.02602989	0.02202631	-0.01047418	0.00897031
0.02	0.01999600	-0.05311731	0.05132736	-0.03924064	0.04708946
0.03	0.02998651	-0.08135889	0.08911485	-0.09392366	0.14546497
0.04	0.03996804	-0.11089284	0.13723154	-0.18817272	0.37882528
0.05	0.04993762	-0.14191637	0.19853676	-0.3488407	0.96345183
0.06	0.05989229	-0.17471238	0.27760603	-0.63830148	2.8459464
0.07	0.06982913	-0.20969134	0.38211536	-1.2643965	7.3313198
0.08	0.07974522	-0.24746	0.52597493	-4.6830624	172.58895
0.09	0.08963770	-0.28893904	0.73832306	0.09914477	26.066958
0.10	0.09950372	-0.33557622	1.1055791	-3.4554085	2.2631308
0.11	0.10934048	-0.38975965	2.7622215	-101.28858	6954.7987
0.12	0.11914522	-0.45569145	1.2805118	-31.530651	-198.73051
0.13	0.12891523	-0.54145696	2.6523173	-35.509464	406.54300
0.14	0.13864784	-0.66476338	5.1130909	-83.609450	1581.1100
0.15	0.14834045	-0.87312659	11.844410	-307.65212	9693.8945
0.16	0.1579905	-1.3516343	43.829227	-2561.5455	183953.19
0.17	0.16759549	-4.2188500	1173.6956	-598387.12	3.80304840×10^8
0.18	0.177153	2.1418255	-41.159648	3581.7333	-454958.98
0.19	0.18666065	0.61071792	12.872215	12.113393	-2462.2706
0.20	0.19611614	0.2351988	7.1665406	44.350811	-72.324822
0.30	0.28734789	-0.37867602	0.98503149	1.8025725	5.8931889
0.40	0.37139068	-0.51715828	0.52418984	0.79232382	0.41160951
0.50	0.4472136	-0.5597459	0.23038891	0.69014814	0.07257866
0.60	0.51449576	-0.55020825	0.02038324	0.53356119	0.27891669
0.70	0.57346234	-0.51417099	-0.10821692	0.34347031	0.34737635
0.80	0.62469505	-0.46755319	-0.17463699	0.18619835	0.29128276
0.90	0.66896473	-0.41919642	-0.20141601	0.07879986	0.20314523
1.00	0.70710678	-0.37342997	-0.20570353	0.01287290	0.12721316
1.10	0.73994007	-0.33202671	-0.19838512	-0.0247471	0.07303315
1.20	0.76822128	-0.29543502	-0.18575195	-0.04470894	0.03751262
1.30	0.79262399	-0.26347544	-0.17123161	-0.05416609	0.01524089
1.40	0.81373347	-0.23570866	-0.15659797	-0.05756331	0.00167726
1.50	0.83205029	-0.21161961	-0.14270895	-0.05756365	-0.00636027
1.60	0.8479983	-0.19070359	-0.12992949	-0.05573405	-0.01094845
1.70	0.86193422	-0.17250244	-0.11836672	-0.05298648	-0.01340201
1.80	0.87415728	-0.15661589	-0.10799852	-0.04984618	-0.01454328
1.90	0.88491822	-0.14270141	-0.09874277	-0.04661101	-0.01488493
2.00	0.89442719	-0.13046906	-0.09049408	-0.04344527	-0.01474681

Table 9: Expansion coefficients of $r_{\mathrm{p}}/r_{\mathrm{e}}$ for various values of ξ_{s} with $\gamma_i = 0$, $i > 2$.

[10] D. Petroff & R. Meinel. Post-Newtonian Approximation of the Rigidly Rotating Disc of Dust to Arbitrary Order. *Phys. Rev. D*, **63**, 064012 (2001).

[11] M. Ansorg, A. Kleinwächter & R. Meinel. Highly Accurate Calculation of Rotating Neutron Stars. *Astron. & Astrophys.*, **381**, L49 (2002).

[12] M. Ansorg, A. Kleinwächter & R. Meinel. Highly Accurate Calculation of Rotating Neutron Stars: Detailed Description of the Numerical Methods. *Astron. & Astrophys.*, **405**, 711 (2003).

[13] N. Stergioulas. Rotating Stars in Relativity, Living Rev. Relativity (2003). Online Publication: Referenced on June 21, 2003, http://www.livingreviews.org/lrr-2003-3.

[14] S. Bonazzola, E. Gourgoulhon & J.A. Marck. Numerical approach for high precision 3-D relativistic star model. *Phys. Rev. D*, **58**, 104020 (1998).

[15] N. Stergioulas & J.L. Friedman. Comparing Models of Rapidly Rotating Relativistic Stars Constructed by Two Numerical Methods. *Astrophys. J.*, **444**, 306 (1995).

[16] H. Komatsu, Y. Eriguchi & I. Hachisu. Rapidly Rotating General Relativistic Stars. I - Numerical Method and its Application to Uniformly Rotating Polytropes. *Mon. Not. R. Astron. Soc.*, **237**, 355 (1989).

[17] H. Komatsu, Y. Eriguchi & I. Hachisu. Rapidly rotating general relativistic stars. II - Differentially rotating polytropes. *Mon. Not. R. Astron. Soc.*, **239**, 153 (1989).

[18] T. Damour, B.R. Iyer & B.S. Sathyaprakash. Improved Filters for Gravitational Waves from Inspiralling Compact Binaries. *Phys. Rev. D*, **57**, 885 (1998).

[19] G. Neugebauer & R. Meinel. General Relativistic Gravitational Field of a Rigidly Rotating Disk of Dust: Solution in Terms of Ultraelliptic Functions. *Phys. Rev. Lett.*, **75**, 3046 (1995).

[20] G. Neugebauer & R. Meinel. Progress in Relativistic Gravitational Theory using the Inverse Scattering Method. *J. Math. Phys*, **44, no. 8**, 3407 (2003).